# Practice Final Exam
# Math 214
(the actual final will be a little shorter)

1.(20 pts) Test the series for convergence or divergence

a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n^2+n+2}{2n^2+4n+7}\right)^n$

**Solution.**

Root test.

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{3n^2+n+2}{2n^2+4n+7}=\frac{3}{2}>1.$$

Divergent.

b) $\displaystyle\sum_{n=1}^{\infty}\frac{e^n(n+1)^2}{n!}$

**Solution.**

Ratio test.

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{e^{n+1}(n+2)^2}{(n+1)!}\frac{n!}{e^n(n+1)^2}$$

$$=\lim_{n\to\infty}\frac{e}{n+1}\frac{(n+2)^2}{(n+1)^2}=0<1.$$

Convergent.

c) $\displaystyle\sum_{n=1}^{\infty}\frac{\sin^2 n}{n^2+1}$

**Solution.**

Comparison test.

$$\frac{\sin^2 n}{n^2+1}\leq\frac{1}{n^2+1}\leq\frac{1}{n^2}.$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent as a $p$-series with $p>2$, the original series is also convergent.

d) $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$

**Solution.**

The $n$th-term test for divergence.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty} e^{\frac{\ln\left(1+\frac{1}{n}\right)}{1/n}}$$

$$= \lim_{n\to\infty} e^{\frac{-1/n^2}{\left(1+\frac{1}{n}\right)(-1/n^2)}} = e \neq 0.$$

Divergent.

2.(10 pts) Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1}{n\ln n}.$$

**Solution.**

$\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1}{n\ln n}$ is an alternating series. $u_n = \frac{1}{n\ln n}$ is a decreasing sequence whose limit is zero. Therefore the series is convergent by the alternating series test.

Now consider

$$\sum_{n=2}^{\infty}|a_n| = \sum_{n=2}^{\infty}\frac{1}{n\ln n}.$$

Let $f(x) = \frac{1}{x\ln x}$. Clearly, $f$ is continuous, positive, decreasing function on $[2,\infty)$. Use the integral test.

$$\int_2^{\infty} f(x)dx = \int_2^{\infty}\frac{1}{x\ln x}dx = \ln\ln x\Big|_2^{\infty} = \infty.$$

Therefore, $\sum_{n=2}^{\infty}|a_n|$ is divergent.

So, the original series is conditionally convergent.

3.(10 pts) Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty}\frac{(x+1)^n}{\sqrt[3]{n}\ 3^n}$.

**Solution.**

We will use the ratio test.

$$\lim_{n\to\infty}\frac{|x+1|^{n+1}}{\sqrt[3]{n+1}3^{n+1}}\frac{\sqrt[3]{n}\ 3^n}{|x+1|^n} = \lim_{n\to\infty}\frac{|x+1|}{3}\sqrt[3]{\frac{n}{n+1}} = \frac{|x+1|}{3}$$

The series converges if the latter limit is less than 1. That is,

$$|x+1| < 3,$$

which means that $-3 < x+1 < 3$ or $x \in (-4, 2)$. Thus, the radius of convergence equals $R = 3$.

Now test the endpoints of the interval $(-4, 2)$.

If $x = -4$, then the series becomes

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{\sqrt[3]{n}\; 3^n} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}.$$

The latter is an alternating series. It converges by the alternating series test, since $u_n = 1/\sqrt[3]{n}$ is a decreasing sequence tending to zero.

If $x = 2$, then the series becomes

$$\sum_{n=0}^{\infty} \frac{3^n}{\sqrt[3]{n}\; 3^n} = \sum_{n=0}^{\infty} \frac{1}{\sqrt[3]{n}}.$$

This is a $p$-series with $p = 1/3$. Thus, divergent.

Therefore, the interval of convergence of the original power series is $[-4, 2)$.

4.(10 pts) Find the Taylor series for the function $f(x) = x^{-2}$ at $x = 1$.

**Solution.**

The Taylor series for a function $f$ centered at $x = 1$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n.$$

Therefore, we need to compute the derivatives of all orders of the given function at $x = 1$.

$$\begin{aligned} f'(x) &= (-2)x^{-3}, \\ f''(x) &= (-2)(-3)x^{-4}, \\ f'''(x) &= (-2)(-3)(-4)x^{-5}, \\ &\cdots \end{aligned}$$

$$f^{(n)}(x) = (-2)(-3)(-4)\cdots(-n)(-n-1)x^{-n-2} = (-1)^n(n+1)!\, x^{-n-2}.$$

Thus, for all $n \geq 0$,

$$f^{(n)}(1) = (-1)^n(n+1)!.$$

The Taylor series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)!}{n!}(x-1)^n = \sum_{n=0}^{\infty} (-1)^n(n+1)(x-1)^n.$$

5.(20 pts) a) Find the slope of the tangent line to the curve $r = 1 + \cos\theta$ at the point where $\theta = \pi/2$.

b) Find the area of the region that lies inside the curve $r = 1 + \cos\theta$ and outside the curve $r = 1$.

**Solution.**

a) Using the relation between polar and Cartesian coordinates

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we get

$$x = (1+\cos\theta)\cos\theta = \cos\theta + \cos^2\theta,$$
$$y = (1+\cos\theta)\sin\theta = \sin\theta + \cos\theta\sin\theta.$$

Now differentiate with respect to $\theta$.

$$\frac{dx}{d\theta} = -\sin\theta - 2\cos\theta\sin\theta,$$

$$\frac{dy}{d\theta} = \cos\theta - \sin^2\theta + \cos^2\theta.$$

The slope is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta - \sin^2\theta + \cos^2\theta}{-\sin\theta - 2\cos\theta\sin\theta}$$

Now substitute the given value $\theta = \pi/2$.

$$\left.\frac{dy}{dx}\right|_{\theta=0} = 1.$$

6.(10 pts) Find the length of the curve $r = \cos^3(\theta/3)$, $0 \le \theta \le \pi/4$.

**Solution.**

$$L = \int_0^{\pi/4} \sqrt{\cos^6(\theta/3) + (-3\sin(\theta/3)\frac{1}{3}\cos^2(\theta/3))^2 d\theta}$$

$$= \int_0^{\pi/4} \sqrt{\cos^6(\theta/3) + \sin^2(\theta/3)\cos^4(\theta/3) d\theta}$$

$$= \int_0^{\pi/4} \cos^2(\theta/3)\sqrt{\sin^2(\theta/3) + \cos^2(\theta/3) d\theta}$$

$$= \int_0^{\pi/4} \cos^2(\theta/3)d\theta = \frac{1}{2}\int_0^{\pi/4} (1 + \cos(2\theta/3))d\theta$$

$$= \frac{1}{2}(\theta + \frac{3}{2}\sin(2\theta/3))\Big|_0^{\pi/4} = \frac{\pi}{8} + \frac{3}{4}\sin(\pi/6) = \frac{\pi+3}{8}.$$

7.(30 pts) a) Find the area of the triangle with vertices $P_1(2,-1,3)$, $P_2(4,0,3)$, and $P_3(3,-2,4)$.

b) Find an equation of the plane passing through these points.

c) Find parametric equations of the line passing through $P_1$ and perpendicular to the plane in part (b).

**Solution.**

a) $\overrightarrow{P_1P_2} = \langle 2,1,0\rangle$, $\overrightarrow{P_1P_3} = \langle 1,-1,1\rangle$.

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}.$$

$$|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}| = \sqrt{14}.$$

The area of the triangle: $\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{14}/2$.

b) The plane passing through these points is

$$(x-2) - 2(y+1) - 3(z-3) = 0,$$

$$x - 2y - 3z + 5 = 0.$$

c) The line is

$$\begin{aligned} x &= 2 + t, \\ y &= -1 - 2t, \qquad t \in \mathbb{R}. \\ z &= 3 - 3t, \end{aligned}$$

8.(10 pts) Find the vector projection of $\mathbf{b} = \langle 6,2,-4\rangle$ onto $\mathbf{a} = \langle 2,-1,-2\rangle$ and the scalar component of $\mathbf{b}$ in the direction of $\mathbf{a}$.

**Solution.**

Vector projection: $\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a} = \dfrac{18}{9}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$.

Scalar component: $\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{18}{3} = 6$.

9.(10 pts) Find the angle between the planes $x+y+3=0$ and $x+2y+2z-1=0$.

**Solution.**

The angle between the planes is the (acute) angle between their normals: $\mathbf{n}_1 = \mathbf{i} + \mathbf{j}$, $\mathbf{n}_2 = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{3}{\sqrt{23}} = \frac{1}{\sqrt{2}}.$$

Therefore, $\theta = \pi/4$.

10.(10 pts) Find the distance from the point $P(2,-1,1)$ to the plane $3x+y-5z+1=0$.

**Solution.**

$$D = \frac{|3\cdot 2 + 1\cdot(-1) - 5\cdot 1 + 1|}{\sqrt{3^2+1^2+(-5)^2}} = \frac{1}{\sqrt{35}}.$$

11.(10 pts) Find parametric equations for the line that is tangent to the curve $\mathbf{r}(t) = \ln(1+t)\mathbf{i} + (1+t)\mathbf{j} - \sin t\mathbf{k}$ at $t=0$.

**Solution.**

$$\mathbf{r}(0) = \mathbf{j},$$

$$\frac{d\mathbf{r}}{dt} = \frac{1}{1+t}\mathbf{i} + \mathbf{j} - \cos t\,\mathbf{k},$$

$$\frac{d\mathbf{r}}{dt}(0) = \mathbf{i} + \mathbf{j} - \mathbf{k}.$$

Tangent line:

$$\begin{array}{rcl} x &=& \tau, \\ y &=& 1+\tau, \\ z &=& -\tau, \end{array} \qquad -\infty < \tau < \infty.$$

12.(10 pts) Find the length of the curve $r(t) = (\sin t - t\cos t)\mathbf{i} + (\cos t + t\sin t)\mathbf{j} + t^2\ \mathbf{k}$, $0 \le t \le 1$.

**Solution.**

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (\cos t - \cos t + t\sin t)\mathbf{i} + (-\sin t + \sin t + t\cos t)\mathbf{j} + 2t\mathbf{k}$$

$$= t\sin t\mathbf{i} + t\cos t\mathbf{j} + 2t\mathbf{k}$$

$$|\mathbf{v}| = \sqrt{t^2\sin^2 t + t^2\cos^2 t + 4t^2}$$

$$= \sqrt{5t^2} = \sqrt{5}\ t.$$

$$L = \int_0^1 |\mathbf{v}|dt = \sqrt{5}\int_0^1 tdt = \frac{\sqrt{5}}{2}\ t^2\Big|_0^1 = \frac{\sqrt{5}}{2}.$$

13.(10 pts) Find the curvature of the curve $r(t) = (\sin t - t\cos t)\mathbf{i} + (\cos t + t\sin t)\mathbf{j} + \mathbf{k}$ at the point where $t = 2$.

**Solution.**

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (\cos t - \cos t + t\sin t)\mathbf{i} + (-\sin t + \sin t + t\cos t)\mathbf{j}$$
$$= t\sin t\mathbf{i} + t\cos t\mathbf{j}$$
$$|\mathbf{v}| = \sqrt{t^2\sin^2 t + t^2\cos^2 t} = t.$$

$$\mathbf{T} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\frac{d\mathbf{T}}{dt} = \cos t\mathbf{i} - \sin t\mathbf{j}$$

$$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} = 1$$
$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{t}.$$

$$\kappa(2) = \frac{1}{2}.$$

14.(10 pts) Find the limit or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}.$$

**Solution.**

Use sandwich theorem.

$$0 \le \left|\frac{xy^2}{x^2+y^2}\right| \le \left|\frac{xy^2}{y^2}\right| = |x|.$$

Since $|x| \to 0$, as $(x,y) \to (0,0)$, we get

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0.$$

15.(10 pts) Find the derivative of the function

$$f(x,y,z) = x^2 - 2xy + xz + z^2 + 2x - y$$

at $P_0(1,1,1)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. In what direction is the derivative of $f$ at $P_0$ maximal? Find the derivative in this direction.

**Solution.** Since $\mathbf{v}$ is not unit, we will find a unit vector in the direction of $\mathbf{v}$.

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}.$$

$$\nabla f = (2x - 2y + z + 2)\mathbf{i} + (-2x - 1)\mathbf{j} + (x + 2z)\mathbf{k},$$

$$\nabla f\Big|_{(1,1,1)} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}.$$

Therefore,

$$(D_{\mathbf{u}}f)\Big|_{(1,1,1)} = \nabla f\Big|_{(1,1,1)} \cdot \mathbf{u} = 5.$$

The derivative is maximal in the direction of $\nabla f\Big|_{P_0}$, which is

$$\frac{1}{\sqrt{3}}\,\mathbf{i} - \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k}.$$

The derivative in this direction is $|\nabla f|\Big|_{P_0} = 3\sqrt{3}$.

16.(10 pts) Find an equation of the tangent plane to the surface $x^2+3y^2+2z^2 = 12$ at the point $P_0(1, 1, 2)$.

**Solution.**

Let $f(x, y, z) = x^2 + 3y^2 + 2z^2 - 12 = 0$. Then

$$\nabla f = 2x\ \mathbf{i} + 6y\ \mathbf{j} + 4z\ \mathbf{k}.$$

$$\nabla f\Big|_{P_0} = 2\ \mathbf{i} + 6\ \mathbf{j} + 8\ \mathbf{k}.$$

The tangent plane is given by the equation:

$$2(x - 1) + 6(y - 1) + 8(z - 2) = 0,$$

$$2x + 6y + 8z - 24 = 0.$$

17.(10 pts) Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = 4x^2 - x^3 + y^2 + 2xy.$$

**Solution.**

$$f_x = 8x - 3x^2 + 2y = 0$$

$$f_y = 2y + 2x = 0$$

From the second equation, $y = -x$. Substitute this into the first equation.

$$6x - 3x^2 = 0,$$

$$3x(2-x)=0,$$
$$x=0 \text{ or } x=2.$$

We get two critical points $(0,0)$ and $(2,-2)$.

Now find $D = f_{xx}f_{yy} - f_{xy}^2$.

$$f_x x = 8-6x, \qquad f_{yy} = 2, \qquad f_{xy} = 2.$$

So, $D = (8-6x)2 - 4 = -12x + 12$.

Point $(0,0)$: $D\Big|_{(0,0)} = 12 > 0$, $f_{xx}\Big|_{(0,0)} = 8 > 0$. Local minimum at $(0,0)$. $f(0,0) = 0$.

Point $(2,-2)$: $D\Big|_{(2,-2)} = -12 < 0$. Saddle point at $(2,-2)$.

18.(10 pts) Find the absolute maximum and minimum values of the function $f(x,y) = x^2 + xy + y^2 - 6x$ on the rectangular region $0 \le x \le 5$, $-3 \le y \le 3$.

**Solution.**

First find the critical points.

$$f_x = 2x + y - 6 = 0$$

$$f_y = x + 2y = 0$$

We get $x = -2y$, then

$$-4y + y - 6 = 0,$$

$$y = -2,$$
$$x = 4.$$

The point $(4,-2)$ belongs to the rectangle. $f(4,-2) = 16-8+4-24 = -12$.

Now consider the sides of the rectangle.

i) $x = 0$, $-3 \le y \le 3$.
$$f(0,y) = y^2.$$

Critical point at $y = 0$. So, we get the point $(0,0)$. $f(0,0) = 0$. Endpoints of this side: $(0,-3)$ and $(0,3)$. $f(0,-3)$ =9, $f(0,3) = 9$.

ii) $y = 3$, $0 \le x \le 5$.

$$f(x,3) = x^2 - 3x + 9.$$

$\frac{d}{dx}(x^2 - 3x + 9) = 2x - 3$. Critical point at $x = 3/2$. So, we get the point $(3/2, 3)$. $f(3/2,3) = 9/4$. Endpoints of this side: $(0,3)$ and $(5,3)$. $f(5,3)$ =19, the value at the other endpoint was computed above.

iii) $x = 5$, $-3 \leq y \leq 3$.

$$f(5, y) = y^2 + 5y - 5.$$

$\frac{d}{dy}(y^2+5y-5) = 2y+5$. Critical point at $y = -5/2$. So, we get the point $(5, -5/2)$. $f(5, -5/2) = -45/4$. Endpoints of this side: $(5, -3)$ and $(5, 3)$. $f(5, -3)$ = -11, the value at the other endpoint was computed above.

iv) $y = -3$, $0 \leq x \leq 5$.

$$f(x, -3) = x^2 - 9x + 9.$$

$\frac{d}{dx}(x^2 - 9x + 9) = 2x - 9$. Critical point at $x = 9/2$. So, we get the point $(9/2, 3)$. $f(9/2, 3) = -45/4$. Endpoints of this side: $(0, 3)$ and $(5, 3)$. The values at the endpoints were computed above.

Now analyze all the candidates. The absolute maximum is 19, achieved at $(5, 3)$. The absolute minimum is -12, achieved at $(4, -2)$.

19.(10 pts) Find the maximum and minimum values of $f(x, y) = x^2 + y^2$ subject to the constraint $x^2 + xy + y^2 = 1$.

**Solution.**

$$f_x = 2x, \qquad f_y = 2y$$
$$g(x, y) = x^2 + xy + y^2 - 1,$$
$$g_x = 2x + y, \qquad g_y x + 2y$$

Then $\nabla f = \lambda \nabla g$ gives

$$2x = \lambda(2x + y)$$
$$2y = \lambda(x + 2y)$$

From the 1st equation,

$$2x(1 - \lambda) = \lambda y$$

So, $y = 2x\frac{1-\lambda}{\lambda}$. Put this into the second equation.

$$4x\frac{1 - \lambda}{\lambda} = \lambda x + 4x(1 - \lambda)$$
$$4x(1 - \lambda) = \lambda^2 x + 4x(1 - \lambda)\lambda$$
$$x[4(1 - \lambda) - \lambda^2 - 4(1 - \lambda)\lambda] = 0$$
$$x = 0 \quad \text{or} \quad 3\lambda^2 - 8\lambda + 4 = 0$$
$$\lambda = 2 \text{ or } \frac{2}{3}$$

If $x = 0$, then $y = 0$ as well, but $(0, 0)$ does not satisfy $x^2+xy+y^2 = 1$.

Now consider $\lambda = 2$. Then $y = -x$. Putting into $x^2 + xy + y^2 = 1$, we see

$$x^2 - x^2 + x^2 = 1$$

So $x = \pm 1$, and and since $y = -x$, we get the points $(1, -1)$, $(-1, 1)$.

$f(1, -1) = 2$, $f(-1, 1) = 2$.

Now consider $\lambda = 2/3$. Then $y = x$. Putting into $x^2 + xy + y^2 = 1$, we see

$$x^2 + x^2 + x^2 = 1$$

So $x = \pm 1/\sqrt{3}$, and and since $y = x$, we get the points $(1/\sqrt{3}, 1/\sqrt{3})$, $(-1/\sqrt{3}, -1/\sqrt{3})$.

$f(1/\sqrt{3}, 1/\sqrt{3}) = 2/3$, $f(-1/\sqrt{3}, -1/\sqrt{3}) = 2/3$.

So, the minimum of $f$ is $2/3$, achieved at $(1/\sqrt{3}, 1/\sqrt{3})$ and $(-1/\sqrt{3}, -1/\sqrt{3})$.

The maximum of $f$ is 2, achieved at $(1, -1)$ and $(-1, 1)$.

20.(10 pts) (**BONUS**) A function $f(x, y)$ is *homogeneous of degree* $n$ ($n$ a non-negative integer) if $f(tx, ty) = t^n f(x, y)$ for all $t$, $x$, and $y$. For such a function (sufficiently differentiable), prove that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y).$$

**Solution.**

Differentiate $f(tx, ty) = t^n f(x, y)$ with respect to $t$.

$$f_x(tx, ty)x + f_y(tx, ty)y = nt^{n-1}f(x, y).$$

Now set $t = 1$.

$$f_x(x, y)x + f_y(x, y)y = nf(x, y).$$